Exam Introduction to Logic (2017-18)
Model Answers

1: Translating to propositional logic (10 points)

Translation key:
L: they like loud music
J: they play jazz
D: they miss a drummer
S: they miss a saxophonist.

a. $L \to J$
   They like loud music only if they play jazz.

b. $J \vee (D \wedge S)$
   They play jazz unless they miss a drummer and a saxophonist.
   Other correct options: $\neg J \to (D \wedge S)$ or $\neg(D \wedge S) \to J$.

2: Translating to first-order logic (10 points)

Translation key:
a: Aaron
b: Barbara
$C(x, y)$: x cooks for y
$F(x, y)$: y is x's friend—NOTE: "x is y's friend" also correct
$V(x)$: x is vegetarian

a. $\forall x((F(a, x) \wedge C(x, a)) \to C(a, x))$
   Aaron cooks for all his friends who cook for him.

b. $(\exists x C(b, x) \to C(b, a)) \wedge V(a) \wedge \forall x((F(b, x) \wedge V(x)) \to x = a)$
   If there is someone for whom Barbara cooks, then she cooks for Aaron, who is the only vegetarian among Barbara's friends.
   Another correct option: $(\exists x C(b, x) \to C(b, a)) \wedge V(a) \wedge \neg\exists x(F(b, x) \wedge V(x) \wedge x \neq a)$

3: Formal proofs (20 points)

a.

1.	$\neg(A \to B)$	
2.	$B \vee C$	
3.	$\quad B$	
4.	$\quad\quad A$	
5.	$\quad\quad B$	Reit: 3
6.	$\quad A \to B$	$\to$ Intro: 4–5
7.	$\quad \bot$	$\bot$ Intro: 6, 1
8.	$\neg B$	$\neg$ Intro: 3-7
9.	$\quad \neg C$	
10.	$\quad\quad B$	
11.	$\quad\quad \bot$	$\bot$ Intro: 10, 8
12.	$\quad\quad C$	
13.	$\quad\quad \bot$	$\bot$ Intro: 12, 9
14.	$\quad \bot$	$\vee$ Elim: 2, 10-11, 12-13
15.	$\quad \neg\neg C$	$\neg$ Intro: 9-14
16.	C	$\neg$ Elim: 15

b.

1.	$\quad \neg(A \wedge B) \vee C$	
2.	$\quad\quad C$	
3.	$\quad\quad\quad A$	
4.	$\quad\quad\quad\quad B$	
5.	$\quad\quad\quad\quad C$	Reit: 2
6.	$\quad\quad\quad B \to C$	$\to$ Intro: 4-5
7.	$\quad\quad A \to (B \to C)$	$\to$ Intro: 3-6
8.	$\quad\quad \neg(A \wedge B)$	
9.	$\quad\quad\quad A$	
10.	$\quad\quad\quad\quad B$	
11.	$\quad\quad\quad\quad A \wedge B$	$\wedge$ Intro: 9,10
12.	$\quad\quad\quad\quad \bot$	$\bot$ Intro: 11, 8
13.	$\quad\quad\quad\quad C$	$\bot$ Elim: 12
14.	$\quad\quad\quad B \to C$	$\to$ Intro: 10-13
15.	$\quad\quad A \to (B \to C)$	$\to$ Intro: 9-14
16.	$\quad A \to (B \to C)$	$\vee$ Elim: 1, 8-15, 2–7
17.	$(\neg(A \wedge B) \vee C) \to (A \to (B \to C))$	$\to$ Intro: 1-16

c.

1. $\exists x P(x) \to Q$

2. $\boxed{\text{a}}$

3. $P(a)$

4. $\exists x P(x)$ — $\exists$ Intro: 3

5. Q — $\to$ Elim: 1,4

6. $P(a) \to Q$ — $\to$ Intro: 3-5

7. $\forall x(P(x) \to Q)$ — $\forall$ Intro: 2-6

8. $\forall x(P(x) \to Q)$

9. $\exists x P(x)$

10. $\boxed{\text{a}} P(a)$

11. $P(a) \to Q$ — $\forall$ Elim: 8

12. Q — $\to$ Elim: 11,10

13. Q — $\exists$ Elim: 9, 10-12

14. $\exists x P(x) \to Q$ — $\to$ Intro: 9-13

$(\exists x P(x) \to Q) \leftrightarrow \forall x(P(x) \to Q)$ — $\leftrightarrow$ Intro: 1-8,9-14

d.

1. $\exists y \forall x R(x,y)$

2. $\boxed{\text{a}}$

3. $\boxed{\text{b}} \forall x R(x,b)$

4. $R(a,b)$ — $\forall$ Elim 3

5. $\exists y R(a,y)$ — $\exists$ Intro: 4

6. $\exists y R(a,y)$ — $\exists$ Elim: 1, 3-5

7. $\forall x \exists y R(x,y)$ — $\forall$ Intro: 2-6

4: Truth tables (10 points)

a.

A	B	C	$(A$	$\to$	$(\neg B$	$\vee$	$C))$	$\to$	$((A$	$\to$	$B)$	$\to$	$\neg$	$(A$	$\wedge$	$\neg C))$
T	T	T	T	T	F	T	T	T	T	T	T	T	T	T	F	F
T	T	F	T	F	F	F	F	T	T	T	T	F	F	T	T	T
T	F	T	T	T	T	T	T	T	T	F	F	T	T	T	F	F
T	F	F	T	T	T	T	F	T	T	F	F	T	F	T	T	T
F	T	T	F	T	F	T	T	T	F	T	T	T	T	F	F	F
F	T	F	F	T	F	F	F	T	F	T	T	T	T	F	F	T
F	F	T	F	T	T	T	T	T	F	T	F	T	T	F	F	F
F	F	F	F	T	T	T	F	T	F	T	F	T	T	F	F	T

The formula is a tautology as witnessed by the green column.

b.

a = b	FrontOf(a, b)	BackOf(a, b)	¬FrontOf(a, b)	$\neg(a = b)$	BackOf(a, b)
T	T	T	F	F	T
T	T	F	F	F	F
T	F	T	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	T	F	F	T	F
F	F	T	T	T	T
F	F	F	T	T	F

The argument is not valid, as witnessed by the green row (both premises are true but the conclusion is false).

Red rows are spurious (first four because the two objects are assumed to be the same and an object cannot be in front of at back of itself; fifth because an object cannot be in front and at back of another object at the same time).

5: Normal forms propositional logic (5 points)

$$
\begin{aligned}
\neg(A \vee B) \to (\neg A \wedge \neg(B \vee C)) &\Leftrightarrow \neg\neg(A \vee B) \vee (\neg A \wedge \neg(B \vee C)) \quad \text{semantics implication}\\
&\Leftrightarrow (A \vee B) \vee (\neg A \wedge \neg(B \vee C)) \quad \text{double negation}\\
&\Leftrightarrow (A \vee B) \vee (\neg A \wedge (\neg B \wedge \neg C)) \quad \text{DeMorgan}\\
&\Leftrightarrow (\neg A \vee (A \vee B)) \wedge ((\neg B \wedge \neg C) \vee (A \vee B)) \quad \text{Dist. } \vee \text{ over } \wedge\\
&\Leftrightarrow (\neg A \vee (A \vee B)) \wedge (\neg B \vee (A \vee B)) \wedge (\neg C \vee (A \vee B)) \quad \text{Dist. } \vee \text{ over } \wedge\\
&\Leftrightarrow (\neg A \vee A \vee B) \wedge (\neg B \vee A \vee B) \wedge (\neg C \vee A \vee B) \quad \text{Brackets removed}
\end{aligned}
$$

6: Normal forms for first-order logic and Horn sentences (10 points)

a.

$$
\begin{aligned}
\neg(\exists x P(x) \vee \forall x \forall y \exists z Q(x,y,z)) &\Leftrightarrow \neg\exists x P(x) \wedge \neg\forall x \forall y \exists z Q(x,y,z) && \text{DeMorgan}\\
&\Leftrightarrow \forall x \neg P(x) \wedge \exists x \neg\forall y \exists z Q(x,y,z) && \neg \text{ over quantifier}\\
&\Leftrightarrow \forall x \neg P(x) \wedge \exists x \exists y \neg\exists z Q(x,y,z) && \neg \text{ over quantifier}\\
&\Leftrightarrow \forall x \neg P(x) \wedge \exists x \exists y \forall z \neg Q(x,y,z) && \neg \text{ over quantifier}\\
&\Leftrightarrow \forall w \neg P(w) \wedge \exists x \exists y \forall z \neg Q(x,y,z) && \text{renaming}\\
&\Leftrightarrow \forall w (\neg P(w) \wedge \exists x \exists y \forall z \neg Q(x,y,z)) && \text{quantifier out}\\
&\Leftrightarrow \forall w \exists x (\neg P(w) \wedge \exists y \forall z \neg Q(x,y,z)) && \text{quantifier out}\\
&\Leftrightarrow \forall w \exists x \exists y (\neg P(w) \wedge \forall z \neg Q(x,y,z)) && \text{quantifier out}\\
&\Leftrightarrow \forall w \exists x \exists y \forall z (\neg P(w) \wedge \neg Q(x,y,z)) && \text{quantifier out}
\end{aligned}
$$

b.

$$\forall x \forall z \forall w (R(x, f(x)) \vee (Q(f(x), z, w) \wedge \neg P(g(x, z, w))))$$

with f (unary), g (ternary) Skolem functions.

c.

A	B	C	D	E	$\neg A$	$\wedge$	B	$\wedge($	C	$\vee$	$\neg B$	$)\wedge($	$\neg C$	$\vee$	$(\neg D$	$\vee$	$E)$	$)\wedge($	$\neg B$	$\vee$	$\neg E)$
F	T	T	F	F	T		T		T	T	F		F	T	T	T	F		F	T	T

Order of assignments for the algorithm: $B :=$ T, $C :=$ T, $E :=$ F, $D :=$ F, $A :=$ F. All conjuncts can be made true. The formula is therefore satisfiable.

7: Tarski's World (10 points)

a. $\forall x(\mathsf{Tet}(x) \leftrightarrow \mathsf{Large}(x))$

b. (i) $\mathsf{SameShape}(\mathsf{a}, \mathsf{b}) \rightarrow (\neg\mathsf{Larger}(\mathsf{a}, \mathsf{b}) \wedge \neg\mathsf{Larger}(\mathsf{b}, \mathsf{a}))$ is **true**.

(ii) $\forall x(\mathsf{Dodec}(x) \rightarrow \mathsf{Small}(x))$ is **true**.

(iii) $\exists x \exists y \exists z(\mathsf{SameShape}(x, y) \wedge \mathsf{SameShape}(y, z) \wedge \mathsf{Between}(x, y, z))$ is **false**.

(iv) $\forall x \neg\exists y((x \neq y \wedge \mathsf{SameShape}(x, y)) \rightarrow (\mathsf{FrontOf}(y, x) \rightarrow \mathsf{Larger}(y, x)))$ is **false**. To appreciate that consider the formula is equivalent to:

$$\forall x \neg\exists y(x = y \vee \neg\mathsf{SameShape}(x, y) \vee \neg\mathsf{FrontOf}(y, x) \vee \mathsf{Larger}(y, x))$$

It suffices to select an assignment (e.g., $x = y = a$) for which one of the above disjuncts hold (e.g., $x = y$).

c.

$$\forall \mathsf{x} \forall \mathsf{y}((\mathsf{FrontOf}(\mathsf{y}, \mathsf{x}) \wedge \mathsf{SameColumn}(\mathsf{x}, \mathsf{y})) \rightarrow \exists \mathsf{z}(\mathsf{SameShape}(\mathsf{z}, \mathsf{y}) \wedge \mathsf{FrontOf}(\mathsf{z}, \mathsf{y}) \wedge \neg\mathsf{Larger}(\mathsf{z}, \mathsf{x})))$$

Several options possible to make the above formula false (list non exhaustive):

(i) c can be moved (at least) 1 cell down. After this move, e and c are on the same column, c is in front of e, but there exists no object which has the same shape of c and it is in front of c.

(ii) c can be moved 1 cell left, or 5 cells right;

(iii) d can be moved 2 cells left;

(iv) a can be moved 1 cell right;

(v) f can be moved 5 cells left.

8: Translating function symbols (5 points)

a. $Morefamous(firstmaster(r), firstmaster(m)) \wedge \exists x(Morefamous(firstmaster(x), firstmaster(r)) \wedge Morefamous(firstmaster(x), firstmaster(m)))$

b. $chiefrival(r) = m \leftrightarrow chiefrival(m) = r$

9: Semantics (10 points)

a. $\mathfrak{M} \models \mathsf{Frequent}(x) \to (\mathsf{Infrequent}(y) \to \mathsf{Beats}(x, y))[h]$

By the semantics of $\to$, this is equivalent to $\mathfrak{M} \not\models \mathsf{Frequent}(x)[h]$ or $\mathfrak{M} \models \mathsf{Infrequent}(y) \to \mathsf{Beats}(x, y)[h]$. Again, by the semantics of $\to$, this is equivalent to a disjunction of the following three disjuncts:

(i) $\mathfrak{M} \not\models \mathsf{Frequent}(x)[h]$

(ii) $\mathfrak{M} \not\models \mathsf{Infrequent}(y)[h]$

(iii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h]$

Disjunct (i) corresponds to the statement $[[x]]_h^{\mathfrak{M}} \notin \mathfrak{M}(\mathsf{Frequent})$, that is, Rock $\notin$ {Rock}, which is false.
Disjunct (ii) corresponds to the statement $[[y]]_h^{\mathfrak{M}} \notin \mathfrak{M}(\mathsf{Infrequent})$, that is, Scissors $\notin$ {Scissors}, which is false.
Disjunct (iii) corresponds to the statement $\langle [[x]]_h^{\mathfrak{M}}, [[y]]_h^{\mathfrak{M}} \rangle \in \mathfrak{M}(\mathsf{Beats})$, that is, $\langle \text{Rock}, \text{Scissors} \rangle \in \{\langle \text{Rock, Scissors} \rangle, \langle \text{Scissors, Paper} \rangle, \langle \text{Paper, Rock} \rangle\}$, which is true.

The original statement is therefore true.

b. $\mathfrak{M} \models \exists x(\neg\mathsf{Infrequent}(x) \wedge \mathsf{Beats}(x, a)[h]$

By the semantics of $\exists$, the above is equivalent to: there exists $d \in D$ such that $\mathfrak{M} \models \neg\mathsf{Infrequent}(x) \wedge \mathsf{Beats}(x, a))[h[x/d]]$. By the semantics of $\wedge$ and $\neg$, this is equivalent to: there exists $d \in D$ such that, $\mathfrak{M} \not\models \mathsf{Infrequent}(x)[h[x/d]]$ and $\mathfrak{M} \models \mathsf{Beats}(x, a))[h[x/d]]$. So this statement is true if and only if there exists $d \in \{\text{Rock}, \text{Scissors}, \text{Paper}\}$ such that both the following statements are true:

(i) $\mathfrak{M} \not\models \mathsf{Infrequent}(x)[h[x/d]]$

(ii) $\mathfrak{M} \models \mathsf{Beats}(x, a))[h[x/d]]$

There are three cases to consider, but to make the original sentence true, it suffices to find one which makes both the above statements true. So let d = Paper. Statement (i) corresponds to $[[x]]_{h[x/d]}^{\mathfrak{M}} \notin \mathfrak{M}(\mathsf{Infrequent})$ and therefore to Paper $\notin$ {Scissors}, which is true. Statement (ii) corresponds to $\langle [[x]]^{\mathfrak{M}}_{h[x/d]}, [[a]]^{\mathfrak{M}}_{h[x/d]} \rangle \in \mathfrak{M}(\mathsf{Beats})$, and therefore to $\langle \text{Paper}, \text{Rock} \rangle \in \{\langle \text{Rock, Scissors} \rangle, \langle \text{Scissors, Paper} \rangle, \langle \text{Paper, Rock} \rangle\}$, which is also true.

The original statement is therefore true.

c. $\mathfrak{M} \models \exists x \forall y \mathsf{Beats}(x, y)[h]$

By the semantics of $\exists$, the above statement is equivalent to: there exists $d \in D$ such that $\mathfrak{M} \models \forall y \mathsf{Beats}(x, y)[h[x/d]]$. By the semantics of $\forall$, this is in turn equivalent to: there exists $d \in D$ such that, for all $e \in D$, $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/d][y/e]]$.

There are three cases, at least one of which should make the statement (†) "$forall e \in D$, $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/d][y/e]]$"true, in order for the original statement to be true.

$\boxed{d = \text{Rock}}$ Then † is equivalent to the conjunction of the following three statements:

(i) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Rock}][y/\text{Rock}]]$

(ii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Rock}][y/\text{Scissors}]]$

(iii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Rock}][y/\text{Paper}]]$

Statement (i) corresponds to $\langle [[x]]^{\mathfrak{M}}_{h[x/\text{Rock}][y/\text{Rock}]}, [[y]]^{\mathfrak{M}}_{h[x/\text{Rock}][y/\text{Rock}]} \rangle \in \mathfrak{M}(\mathsf{Beats})$, and is therefore equivalent to $\langle \text{Rock}, \text{Rock} \rangle \in \{\langle \text{Rock, Scissors} \rangle, \langle \text{Scissors, Paper} \rangle, \langle \text{Paper, Rock} \rangle\}$, which is false. This suffices to establish that † is false under the assumption that $d = \text{Rock}$.

$\boxed{d = \text{Scissors}}$ Then † is equivalent to the conjunction of the following three statements:

(i) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Scissors}][y/\text{Rock}]]$

(ii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Scissors}][y/\text{Scissors}]]$

(iii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Scissors}][y/\text{Paper}]]$

Statement (i) corresponds to $\langle [[x]]^{\mathfrak{M}}_{h[x/\text{Scissors}][y/\text{Rock}]}, [[y]]^{\mathfrak{M}}_{h[x/\text{Scissors}][y/\text{Rock}]} \rangle \in \mathfrak{M}(\mathsf{Beats})$, and is therefore equivalent to $\langle \text{Scissors}, \text{Rock} \rangle \in \{\langle \text{Rock, Scissors} \rangle, \langle \text{Scissors, Paper} \rangle, \langle \text{Paper, Rock} \rangle\}$, which is false. This suffices to establish that † is false under the assumption that $d = \text{Scissors}$.

$\boxed{d = \text{Paper}}$ Then † is equivalent to the conjunction of the following three statements:

(i) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Paper}][y/\text{Rock}]]$

(ii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Paper}][y/\text{Scissors}]]$

(iii) $\mathfrak{M} \models \mathsf{Beats}(x, y)[h[x/\text{Paper}][y/\text{Paper}]]$

Statement (ii) corresponds to $\langle [[x]]^{\mathfrak{M}}_{h[x/\text{Paper}][y/\text{Scissors}]}, [[y]]^{\mathfrak{M}}_{h[x/\text{Paper}][y/\text{Scissors}]} \rangle \in \mathfrak{M}(\mathsf{Beats})$, and is therefore equivalent to $\langle \text{Paper}, \text{Scissors} \rangle \in \{\langle \text{Rock, Scissors} \rangle, \langle \text{Scissors, Paper} \rangle, \langle \text{Paper, Rock} \rangle\}$, which is false. This suffices to establish that † is false under the assumption that $d = \text{Paper}$.

In all three cases † is false. Therefore the initial statement is false.

10: Bonus question (10 points)

1. $\forall x \exists y R(x,y) \to \exists y \forall x R(x,y)$	
2. $\neg(\forall x \exists y R(x,y) \lor \neg \forall x \exists y R(x,y))$	
3. $\forall x \exists y R(x,y)$	
4. $\forall x \exists y R(x,y) \lor \neg \forall x \exists y R(x,y)$	$\lor$ Intro: 3
5. $\bot$	$\bot$ Intro: 4,2
6. $\neg \forall x \exists y R(x,y)$	$\neg$ Intro: 3-5
7. $\forall x \exists y R(x,y) \lor \neg \forall x \exists y R(x,y)$	$\lor$ Intro: 7
8. $\bot$	$\bot$ Intro: 7-2
9. $\neg\neg(\forall x \exists y R(x,y) \lor \neg \forall x \exists y R(x,y))$	$\neg$ Intro: 2-8
10. $\forall x \exists y R(x,y) \lor \neg \forall x \exists y R(x,y)$	$\neg$ Elim: 9
11. $\forall x \exists y R(x,y)$	
12. $\exists y \forall x R(x,y)$	$\to$ Elim: 1, 11
13. $\exists y \forall x R(x,y) \lor \exists x \forall y \neg R(x,y)$	$\lor$ Intro: 12
14. $\neg \forall x \exists y R(x,y)$	
15. $\neg \exists x \forall y \neg R(x,y)$	
16. $\boxed{a}$	
17. $\forall y \neg R(a,y)$	
18. $\exists x \forall y \neg R(x,y)$	$\exists$ Intro: 17
19. $\bot$	$\bot$ Intro: 18,15
20. $\neg \forall y \neg R(a,y)$	$\neg$ Intro: 17-19
21. $\neg \exists y R(a,y)$	
22. $\boxed{b}$	
23. $R(a,b)$	
24. $\exists y R(a,y)$	$\exists$ Intro 23
25. $\bot$	$\bot$ Intro: 24, 21
26. $\neg R(a,b)$	$\neg$ Intro: 23-25
27. $\forall y \neg R(a,y)$	$\forall$ Intro: 22-26
28. $\bot$	$\bot$ Intro: 27,21
29. $\neg\neg \exists y R(a,y)$	$\neg$ Intro: 21-28
30. $\exists y R(a,y)$	$\neg$ Elim: 29
31. $\forall x \exists y R(x,y)$	
32. $\bot$	$\bot$ Intro: 32,14
33. $\neg\neg \exists x \forall y \neg R(x,y)$	$\neg$ Intro: 15-32
34. $\exists x \forall y \neg R(x,y)$	$\neg$ Elim: 33
35. $\exists y \forall x R(x,y) \lor \exists x \forall y \neg R(x,y)$	$\lor$ Intro: 33
36. $\exists y \forall x R(x,y) \lor \exists x \forall y \neg R(x,y)$	$\lor$ Elim: 10, 11-13, 14-35
37. $\exists x \forall y \neg R(x,y)$	
38. $\boxed{c}\forall y \neg R(c,y)$	
39. $\forall y \neg R(c,y) \lor \forall y R(y,c)$	$\lor$ Intro: 38
40. $\exists x(\forall y \neg R(x,y) \lor \forall y R(y,x))$	$\exists$ Intro: 39
41. $\exists x(\forall y \neg R(x,y) \lor \forall y R(y,x))$	$\exists$ Elim: 37, 38-41
42. $\exists y \forall x R(x,y)$	
43. $\boxed{d}\forall x R(x,d)$	
44. $\boxed{e}$	
45. $R(e,d)$	$\forall$ Elim: 43
46. $\forall y R(y,d)$	$\forall$ Intro: 44-45
47. $\forall y R(y,d) \lor \forall y \neg R(d,y)$	$\lor$ Intro: 46
48. $\exists x(\forall y R(y,x) \lor \forall y \neg R(x,y))$	$\exists$ Intro: 47
49. $\exists x(\forall y R(y,x) \lor \forall y \neg R(x,y))$	$\exists$ Elim: 42, 43-48
50. $\exists x(\forall y \neg R(x,y) \lor \forall y R(y,x))$	$\lor$ Elim: 36, 42-49, 37-41